Lecture 15

Systems with Sampled-Data and Quantized Output

The approach for resilient feedback designing is provided for a wide range of nonlinear systems with internal uncertainty as well as external constrained disturbances. The associated BMI's, which are demonstrated to be reduced to the system of LMI's, describe the class of stabilizing dynamic feedbacks with a specified linear structure. Under a set of specified linear matrix constraints, the optimal parameters of the feedback controllers actualize the matrix solution to conditional optimization problems. The normal MATLAB packages are used to address this optimization issue. The recommended approach's viability is demonstrated by numerical examples.

15.1 Sampling and quantization

The control community has seen a revived interest in phenomena that are intrinsic to the digital implementation of continuous-time control systems, such as *sampling* and *quantization*, spurred by increasing applications in networked control systems [22-24]. A significant line of study in this field integrates information-theoretical components of the networked control issue (such as channel capacity) and tries to develop a theory that is comparable to the well-known mathematical theory of communication [25]. Following this path has yielded some interesting outcomes. For example, it is now feasible to connect the absolute magnitude of a system's unstable eigenvalues to the minimal channel capacity necessary to stabilize it [26-28]. While theoretically fascinating, the majority of these findings are restricted to linear systems thus far. The topic is framed in a stochastic framework, with the coding and decoding components of the communication channel receiving special attention (see [29] for a coding scheme).

Depending on whether quantization influences the control or output signals, quantization can be viewed as deterministic noise or deterministic perturbation. To deal with the quantization problem, a robust-control technique, such as H_∞ [30] or the sector bound [31], can be used. Again, the majority of the findings obtained with this method are restricted to linear systems. In this regard, the present lecture can be seen as an extension of the work presented in [32] to the case nonlinear models when quantization phenomena are present. Notice that the same problem has been considered in [33] where the feedback has built using the state estimates generated by the Luenberger-like filter. Here we follow [1], [34] and considere the designing of robust full-order dynamic feedback as in lecture 13.

15.2 System Description and problem formulation

Consider the nonlinear system

$$\boxed{\dot{x}(t) = f(t, x(t)) + Bu(t) + \xi_x(t)} \tag{15.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ $(m \leq n)$ and $\xi_x(t) \in \mathbb{R}^n$ are, respectively, the state vector, control input and perturbation at time $t \geq 0$. We use the following model to describe a noisy, sampled and quantized output $y(t)$:

$$\tilde{y}(t) = Cx(t) + \xi_y(t) \tag{15.2}$$

$$\bar{y}(t) = \sum_k \tilde{y}(t_k)\chi(t \in [t_k, t_{k+1})) \tag{15.3}$$

$$y(t) = \pi(\bar{y}(t)) \tag{15.4}$$

The vector $\xi_y(t) \in \mathbb{R}^l$ in (15.2) is the deterministic noise. The symbol $\chi(t \in [t_k, t_{k+1}))$ in (15.3) means thecharacteristic function of the time interval $[t_k, t_{k+1})$, i.e.,

$$\chi(t \in [t_k, t_{k+1})) = \begin{cases} 1 & \text{if} \quad t \in [t_k, t_{k+1}) \\ 0 & \text{if} \quad \text{otherwise,} \end{cases}, \ k = 0, 1, 2, ...; \ t_0 = 0.$$

Thus, $\bar{y}(t) : \mathbb{R}_+ \to \mathbb{R}^l$ is a piecewise constant function which is obtained by sampling and holding $\bar{y}(t)$ at the discrete instants t_k. The actual system output at time t is $y(t) \in \mathbb{R}^l$, and is obtained by quantizing the sampled signal $\bar{y}(t)$. Formally, let $Y \subset \mathbb{R}^l$ be a countable set of possible output values. Then, $\pi : \mathbb{R}^l \to Y$ in (15.4) is defined as a *projection operator*, i.e., as an operator that satisfies

$$\pi \circ \pi\left(\bar{y}(t)\right) \equiv \pi\left(\bar{y}(t)\right).$$

The image of π is a discrete subset of $\mathbb{R}^l$. For example, each component $y_i(t)$ after the projection may take its values on the set of fixed points

$$\left\{y_i^{(1)}, y_i^{(2)}, \ldots\right\}$$

as they are depicted in Fig.15.1.

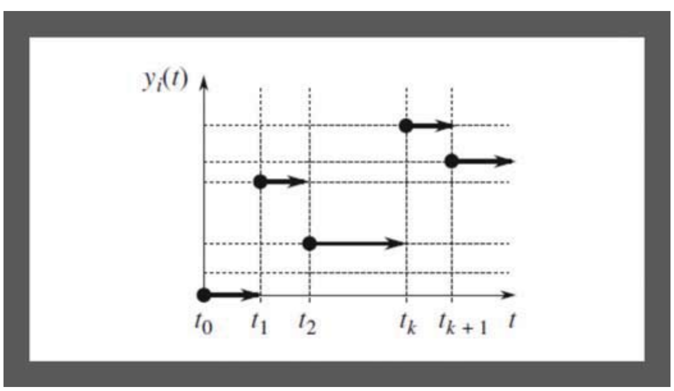

Figure 15.1: Sampled and quantized output signal.

Notice that from (15.2) - (15.4) it follows that for all $t \in [t_k, t_{k+1})$

$$y(t) = \pi(\bar{y}(t)) = \pi\left(Cx(t_k) + \xi_y(t_k)\right) = Cx(t) + \xi_y(t) +$$

$$\pi\left(Cx(t_k) + \xi_y(t_k)\right) - \left[Cx(t) + \xi_y(t)\right] +$$

$$Cx(t_k) + \xi_y(t_k) - \left(Cx(t_k) + \xi_y(t_k)\right) =$$

$$Cx(t) + \xi_y(t) +$$

$$\underbrace{\pi\left(Cx(t_k) + \xi_y(t_k)\right) - \left(Cx(t_k) + \xi_y(t_k)\right)}_{\Delta y'\ (t,t_k)} +$$

$$\underbrace{Cx(t_k) + \xi_y(t_k) - \left[Cx(t) + \xi_y(t)\right]}_{\Delta y''(t,t_k)},$$

and hence, the output signal $y(t)$ may be represented as

$$\boxed{y(t) = Cx(t) + \xi_y(t) + \Delta y(t, t_k)} \tag{15.5}$$

where

$$\boxed{\left.\begin{array}{c} \Delta y(t, t_k) = \Delta y'(t, t_k) + \Delta y''(t, t_k), \\ \\ \Delta y'\ (t, t_k) := \pi\left(Cx(t_k) + \xi_y(t_k)\right) - \left[Cx(t) + \xi_y(t)\right], \\ \Delta y''(t, t_k) := C\left[x(t_k) - x(t)\right] + \left[\xi_y(t_k) - \xi_y(t)\right]. \end{array}\right\}} \tag{15.6}$$

Here $\Delta y'\ (t, t_k)$ may be treated as an error of quantization, and $\Delta y''(t, t_k)$ as an error of sampling.

15.3 Basic Assumptions

Let us now formulate our basic assumptions.

(1) The perturbation and noise are unknown but bounded. More precisely, there are known positive definite matrices $Q_x \in \mathbb{R}^{n\times n}$ and $Q_y \in \mathbb{R}^{l\times l}$ such that

$$2\left\|\xi_x(t)\right\|^2_{Q_x} + \left\|\xi_y(t)\right\|^2_{Q_y} \le 1 \tag{15.7}$$

(here $\|\cdot\|_{Q_x}$ and $\|\cdot\|_{Q_y}$ are weighted norms given by Q_x and Q_y.

(2) The function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is also supposed to be unknown but satisfies the quasi-Lipschitz bound

$$\|f(t,x) - Ax\|_Q^2 \leq \frac{1}{2}\left(\delta + \|x\|_Q^2\right) \text{ for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \qquad (15.8)$$

where $\delta > 0$ is a scalar and $Q > 0$ and A are known $(n \times n)$-dimensional matrices.

(3) The pair (A, B) is controllable and (C, A) is observable.

(4) The sampling intervals need not be regular, but there exists a maximum sampling interval

$$h := \max_k (t_{k+1} - t_k), \; t_{k+1} \geq t_k. \qquad (15.9)$$

(5) The quantization error is bounded, i.e., the positive scalar

$$c := \max_{\bar{y} \in \mathbb{R}^l} \|\pi(\bar{y}) - \pi(\bar{y})\|_{Q_y}^2 \qquad (15.10)$$

is finite.

Notice that (15.8) is not restrictive and comprises a large class of unknown nonlinear functions. Defining the auxiliary function

$$\omega_x(t, x(t)) := \xi_x(t) + f(t, x(t)) - Ax(t) \qquad (15.11)$$

we can rewrite (15.1) and (15.2)–(15.4) in the quasi-linear format

$$\boxed{\left.\begin{array}{c} \dot{x}(t) = Ax(t) + Bu(t) + \omega_x(t, x(t)), \\ \\ y(t) = Cx(t) + \xi_y(t) + \Delta y(t, t_k), \; t \in [t_k, t_{k+1}) \end{array}\right\}} \qquad (15.12)$$

Remark 15.1 *The assumption 3, given above, then becomes to be natural.*

15.4 Feedback Structure

The proposed feedback designing is used here in combination with the classical *full-order linear dynamic output controllers* (see lecture 13) given by

$$\boxed{\left.\begin{array}{c} u(t) = C_r x_r\,(t) + D_r y(t), \\ \\ \dot{x}_r\,(t) = A_r x_r\,(t) + B_r y(t), \\ \\ x_r\,(t) = x_0^r, \end{array}\right\}} \tag{15.13}$$

where

$$x_r \in \mathbb{R}^n,\ A_r \in \mathbb{R}^{n\times n},\ D_r \in \mathbb{R}^{n\times l},\ C_r \in \mathbb{R}^{m\times n}.$$

The control design associated with (15.13) is completely defined by a selection of the matrix

$$\boxed{\Theta := \begin{bmatrix} D_r & C_r \\ B_r & A_r \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+l)}.} \tag{15.14}$$

We call Θ the *dynamic controller matrix*. Introducing the extended vectors

$$\boxed{\left.\begin{array}{c} z(t) := \begin{pmatrix} x^\intercal\,(t) & x_r^\intercal\,(t) \end{pmatrix}^\intercal \in \mathbb{R}^{2n}, \\ \\ \eta\,(t,t_k) := \begin{pmatrix} \omega_x^\intercal(t,x\,(t)) & \xi_y\,(t) & \Delta^\intercal y\,(t,t_k) \end{pmatrix}^\intercal \in \mathbb{R}^{n+2l} \end{array}\right\}} \tag{15.15}$$

we can represent the closed-loop realization of (15.12) under (15.13) as

$$\dot{z}(t) = \begin{bmatrix} A + BD_rC & BC_r \\ B_rC & A_r \end{bmatrix} z\,(t) + \begin{bmatrix} I_{n\times n} & BD_r & BD_r \\ 0_{n\times n} & B_r & B_r \end{bmatrix} \eta\,(t,t_k)$$

or, equivalently, with $z(0) = (x_0, x_0^r)$ and for any$t \in [t_k, t_{k+1})$

$$\boxed{\dot{z}(t) = [A_0 + B_0\Theta C_0]\,z\,(t) + [D_0 + B_0\Theta E_0]\,\eta\,(t,t_k)} \tag{15.16}$$

where

$$\boxed{\left.\begin{array}{c} A_0 = \begin{bmatrix} A & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix},\ B_0 = \begin{bmatrix} B & 0_{n\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix}, \\ C_0 = \begin{bmatrix} C & 0_{l\times n} \\ 0_{n\times n} & I_{n\times n} \end{bmatrix}, \\ D_0 = \begin{bmatrix} I_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix},\ E_0 = \begin{bmatrix} 0_{n\times n} & I_{n\times m} & I_{n\times m} \\ 0_{n\times n} & 0_{n\times m} & 0_{n\times m} \end{bmatrix}. \end{array}\right\}} \tag{15.17}$$

15.5 Stability Analysis

15.5.1 The Lyapunov - Krasovskii functional and its derivative

Since sampling entails delays, instead of a regular Lyapunov function let us use a Lyapunov - Krasovskii functional. More precisely, let $\mathcal{C}^0\left(\mathbb{R},\mathbb{R}^{2n}\right)$ be the space of all continuous functions of mapping $\mathbb{R}$ into $\mathbb{R}^{2n}$, differentiable almost everywhere; let matrices $R \geq 0$ and $P > 0$ be $(2n \times 2n)$-dimensional matrices and let $\alpha > 0$ be a scalar. Following [1], propose the functional $V : \mathbb{R} \times \mathcal{C}^0\left(\mathbb{R},\mathbb{R}^{2n}\right) \to \mathbb{R}_+$, defined as

$$V\left(t, z\left(\cdot\right)\right) := z^{\mathsf{T}}\left(t\right) P z\left(t\right) + h \int\limits_{\theta=-h}^{0} \int\limits_{s=t+\theta}^{t} e^{\alpha(s-t)} \dot{z}^{\mathsf{T}}\left(s\right) R \dot{z}\left(s\right) ds d\theta. \quad (15.18)$$

Direct calculation of its time-derivative gives

$$\left.\begin{array}{c}
\dot{V}\left(t, z\left(\cdot\right)\right) = \\
2z^{\mathsf{T}}\left(t\right) P \dot{z}\left(t\right) - \alpha h \int\limits_{\theta=-h}^{0} \int\limits_{s=t+\theta}^{t} e^{\alpha(s-t)} \dot{z}^{\mathsf{T}}\left(s\right) R \dot{z}\left(s\right) ds d\theta \\
+ h \int\limits_{\theta=-h}^{0} \frac{d}{dt} \int\limits_{s=t+\theta}^{t} e^{\alpha(s-t)} \dot{z}^{\mathsf{T}}\left(s\right) R \dot{z}\left(s\right) ds d\theta = \\
2z^{\mathsf{T}}\left(t\right) P \dot{z}\left(t\right) - \alpha h \int\limits_{\theta=-h}^{0} \int\limits_{s=t+\theta}^{t} e^{\alpha(s-t)} \dot{z}^{\mathsf{T}}\left(s\right) R \dot{z}\left(s\right) ds d\theta \\
+ h \int\limits_{\theta=-h}^{0} \dot{z}^{\mathsf{T}}\left(t\right) R \dot{z}\left(t\right) d\theta + h \int\limits_{\theta=-h}^{0} e^{\alpha\theta} \dot{z}^{\mathsf{T}}\left(t+\theta\right) R \dot{z}\left(t+\theta\right) d\theta \\
= 2z^{\mathsf{T}}\left(t\right) P \dot{z}\left(t\right) - \alpha h \int\limits_{\theta=-h}^{0} \int\limits_{s=t+\theta}^{t} e^{\alpha(s-t)} \dot{z}^{\mathsf{T}}\left(s\right) R \dot{z}\left(s\right) ds d\theta \\
+ h^2 \dot{z}^{\mathsf{T}}\left(t\right) R \dot{z}\left(t\right) - h \int\limits_{s=t-h}^{t} e^{\alpha(s-t)} \dot{z}^{\mathsf{T}}\left(s\right) R \dot{z}\left(s\right) ds
\end{array}\right\} \quad (15.19)$$

By adding and subtracting the term $\alpha z^{\mathsf{T}}\left(t\right) P z\left(t\right)$ to the right-hand side of (15.19) we get

$$\left.\begin{array}{c}
\dot{V}\left(t, z\left(\cdot\right)\right) = 2z^{\mathsf{T}}\left(t\right) P \dot{z}\left(t\right) + \alpha z^{\mathsf{T}}\left(t\right) P z\left(t\right) - \alpha V\left(t, z\left(\cdot\right)\right) \\
+ h^2 \dot{z}^{\mathsf{T}}\left(t\right) R \dot{z}\left(t\right) - h \int\limits_{s=t-h}^{t} e^{\alpha(s-t)} \dot{z}^{\mathsf{T}}\left(s\right) R \dot{z}\left(s\right) ds
\end{array}\right\} \quad (15.20)$$

Then using the upper estimate

$$-h\int_{s=t-h}^{t} e^{\alpha(s-t)}\dot{z}^{\intercal}(s)\,R\dot{z}(s)\,ds \le -e^{-\alpha h}\int_{s=t-h}^{t}\dot{z}^{\intercal}(s)\,R\dot{z}(s)\,ds \le$$

$$-e^{-\alpha h}\int_{s=t-\max\limits_{k}(t_{k+1}-t_k)}^{t}\dot{z}^{\intercal}(s)\,R\dot{z}(s)\,ds \le -e^{-\alpha h}\int_{s=t-(t_{k+1}-t_k)}^{t}\dot{z}^{\intercal}(s)\,R\dot{z}(s)\,ds$$

$$= -e^{-\alpha h}\int_{s=t_k+t-t_{k+1}}^{t}\dot{z}^{\intercal}(s)\,R\dot{z}(s)\,ds \overset{t<t_{k+1}}{=} -e^{-\alpha h}\int_{s=t_k}^{t}\dot{z}^{\intercal}(s)\,R\dot{z}(s)\,ds$$

and the Jensen's inequality (for details see Theorem 16.30 in [5])

$$\int_{s=t_k}^{t}\dot{z}^{\intercal}(s)\,R\dot{z}(s)\,ds = \int_{s=t_k}^{t}\left(R^{1/2}\dot{z}(s)\right)^{\intercal}R^{1/2}\dot{z}(s)\,ds =$$

$$\int_{s=t_k}^{t}\left\|R^{1/2}\dot{z}(s)\right\|^{2}ds \ge \left\|\int_{s=t_k}^{t}R^{1/2}\dot{z}(s)\,ds\right\|^{2} =$$

$$\left(\int_{s=t_k}^{t}\dot{z}(s)\,ds\right)^{\intercal}R\left(\int_{s=t_k}^{t}\dot{z}(s)\,ds\right) = [z(t)-z(t_k)]^{\intercal}R\,[z(t)-z(t_k)]$$

valid for any given $z(\cdot)\in\mathcal{C}^{0}\left(\mathbb{R},\mathbb{R}^{2n}\right)$, $h>0$, $\alpha>0$ and $R>0$, from the identity (15.20) we derive the following differential inequality:

$$\dot{V}(t,z(\cdot)) \le 2z^{\intercal}(t)\,P\dot{z}(t)+\alpha z^{\intercal}(t)\,Pz(t)-\alpha V(t,z(\cdot))$$

$$+h^{2}\dot{z}^{\intercal}(t)\,R\dot{z}(t)-e^{-\alpha h}[z(t)-z(t_k)]^{\intercal}R\,[z(t)-z(t_k)],$$

which may be represented as

$$\boxed{\dot{V}(t,z(\cdot)) \le -\alpha V(t,z(\cdot))+\zeta_1^{\intercal}(t)\,W_1\zeta_1(t),} \tag{15.21}$$

where

$$\boxed{\zeta_1(t)=\begin{pmatrix} z(t)\\ \dot{z}(t)\\ \int_{s=t_k}^{t}\dot{z}(s)\,ds\end{pmatrix}=\begin{pmatrix} z(t)\\ \dot{z}(t)\\ z(t)-z(t_k)\end{pmatrix}} \tag{15.22}$$

and

$$\boxed{W_1 = \begin{bmatrix} \alpha P & P & 0_{2n\times 2n} \\ P & h^2 R & 0_{2n\times 2n} \\ 0_{2n\times 2n} & 0_{2n\times 2n} & -he^{-\alpha h}R \end{bmatrix}.} \tag{15.23}$$

15.5.2 Descriptor form

Now we will refine the bound given in (15.21) by restricting $z(t)$ to the set of solutions of (15.16). To do so, we follow the idea presented in [35, 36] which, originally devised for systems in descriptor form, consists in adding a term (the descriptor term) to the expression (15.21) for $\dot{V}(t, z(\cdot))$. The descriptor term has to be zero for any solution $z(t)$ of the system. In our case, we will add the term

$$\mathcal{D}(t, z(\cdot)) :=$$

$$2\left[z^{\mathsf{T}}(t)\Pi_a + \dot{z}^{\mathsf{T}}(t)\Pi_b\right]\left[(A_0 + B_0\Theta C_0)\, z(t) + (D_0 + B_0\Theta E_0)\,\eta(t, t_k) - \dot{z}(t)\right] = 0,$$

where Π_a and Π_b are $(2n \times 2n)$ matrices. Obviously, $\mathcal{D}$ is zero along the solutions of (15.16). As the result we obtain

$$\left.\begin{aligned} \dot{V}(t, z(\cdot)) &\le -\alpha V(t, z(\cdot)) + \zeta_1^{\mathsf{T}}(t)\, W_1 \zeta_1(t) + \mathcal{D}(t, z(\cdot)) \\ &= -\alpha V(t, z(\cdot)) + \eta^{\mathsf{T}}(t, t_k)\, Q\eta(t, t_k) + \zeta^{\mathsf{T}}(t)\, W_2 \zeta(t) \end{aligned}\right\} \tag{15.24}$$

with

$$\zeta(t) = \begin{pmatrix} \zeta_1^{\mathsf{T}}(t) & \eta^{\mathsf{T}}(t, t_k) \end{pmatrix}^{\mathsf{T}} \in \mathbb{R}^{\varkappa}, \;\; \varkappa = 7n + 2l,$$

$$Q = \begin{bmatrix} Q_x & 0 & 0 \\ 0 & Q_y & 0 \\ 0 & 0 & Q_y \end{bmatrix}$$

and

$$\left.\begin{array}{c}
W_2 = \\
\begin{bmatrix}
W_2^{(1,1)} & W_2^{(1,2)} & 0 & W_2^{(1,4)} \\
\left(W_2^{(1,2)}\right)^\intercal & W_2^{(2,2)} & 0 & W_2^{(2,4)} \\
0 & 0 & -he^{-\alpha h}R & 0 \\
\left(W_2^{(1,4)}\right)^\intercal & \left(W_2^{(2,4)}\right)^\intercal & 0 & -Q
\end{bmatrix}, \\
\\
W_2^{(1,1)} = \alpha P + \Pi_a \left(A_0{+}B_0\Theta C_0\right) + \\
\left(A_0{+}B_0\Theta C_0\right)^\intercal \Pi_a^\intercal \in \mathbb{R}^{2n\times 2n}, \\
W_2^{(1,2)} = P + \Pi_b \left(A_0{+}B_0\Theta C_0\right) \in \mathbb{R}^{2n\times 2n}, \\
W_2^{(1,4)} = \Pi_a \left(D_0{+}B_0\Theta E_0\right) \in \mathbb{R}^{2n\times (n+2m)} \\
W_2^{(2,2)} = h^2 R - \Pi_b {-} \Pi_b^\intercal \in \mathbb{R}^{2n\times 2n}, \\
W_2^{(2,4)} = \Pi_b \left(D_0{+}B_0\Theta E_0\right) \in \mathbb{R}^{2n\times (n+2m)}.
\end{array}\right\} \tag{15.25}$$

Recall that

$$x(t) = Mz(t), M := \begin{bmatrix} I_{n\times n} & 0_{n\times n} \end{bmatrix} \tag{15.26}$$

so that

$$\|x(t)\|_{Q_x}^2 = z^\intercal(t)\left[M^\intercal Q_x M\right] z(t),$$

$$\Delta y''(t, t_k) := C\left[x(t_k) - x(t)\right] + \left[\xi_y(t_k) - \xi_y(t)\right]$$

$$= -CM[z(t) - z(t_k)] + \left[\xi_y(t_k) - \xi_y(t)\right].$$

Notice also that by the assumptions (1) and (2), relations (15.6) and (15.26) becomes

$$\left.\begin{array}{c}
\eta^\intercal(t, t_k)\, Q\eta(t, t_k) = \|\eta(t, t_k)\|_Q^2 = \|\xi_x(t_k)\|_{Q_x}^2 + \\
\left\|\xi_y(t_k)\right\|_{Q_y}^2 + \|\Delta y(t, t_k)\|_{Q_y}^2 \le 2\|\xi_x(t_k)\|_{Q_x}^2 + \\
2\|f(t, x(t)) - Ax\|_{Q_x}^2 + \left\|\xi_y(t_k)\right\|_{Q_y}^2 + \|\Delta y(t, t_k)\|_{Q_y}^2 \\
\le \beta + z^\intercal(t)\left[M^\intercal Q_x M\right] z(t) + \\
\left[z(t) - z(t_k)\right]^\intercal \left[4M^\intercal C^\intercal Q_y CM\right]\left[z(t) - z(t_k)\right]
\end{array}\right\} \tag{15.27}$$

with

$$\beta = 9 + \delta + 2c. \tag{15.28}$$

Finally, substituting (15.27) into (15.24) implies

$$\boxed{\dot{V}\left(t,z\left(\cdot\right)\right) \leq -\alpha V\left(t,z\left(\cdot\right)\right) + \beta + \zeta^{\intercal}\left(t\right) W \zeta\left(t\right)} \tag{15.29}$$

with

$$\boxed{W = W_2 + \begin{bmatrix} M^{\intercal} Q_x M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4M^{\intercal} C^{\intercal} Q_y C M & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}. \tag{15.30}$$

15.6 Main Result

Now we are ready to formulate the main result of this paper, concerning the parametrization of all full-order dynamic feedbacks with the parameter Θ which guaranty the boundedness of any possible trajectory of the considered dynamic system closed by this feedback.

Theorem 15.1 *Under the assumptions (1)–(5), given above, the control system (15.12), closed by the dynamic feedback (15.13) with the parametric matrix Θ, provides the boundedness of all possible trajectories,which asymptotically tend to the attractive ellipsoid $\mathcal{E}\left(0, \frac{\beta}{\alpha} P\right)$, if there exist positive scalar α, matrices Π_a and Π_b, and a positive definite matrix $P = P^{\intercal} > 0$ satisfying the matrix inequality*

$$\boxed{W = W(P, \Pi_a, \Pi_b, \Theta, \alpha) < 0} \tag{15.31}$$

where W is defined by (15.30).

Proof. It follows directly from (15.29) if take into account (15.31) and obtain

$$\dot{V}\left(t,z\left(\cdot\right)\right) \leq -\alpha V\left(t,z\left(\cdot\right)\right) + \beta,$$

which implies

$$V\left(t,z\left(\cdot\right)\right) \leq \frac{\beta}{\alpha} + \left[V\left(0,z\left(0\right)\right) - \frac{\beta}{\alpha}\right] e^{-\alpha t}.$$

Tending $t \to \infty$ leads to

$$\limsup_{t\to\infty} z^{\intercal}\left(t\right) P z\left(t\right) \leq \limsup_{t\to\infty} V\left(t,z\left(\cdot\right)\right) \leq \frac{\beta}{\alpha},$$

or, equivalently, to

$$\limsup_{t\to\infty} z^{\intercal}(t)\left(\frac{\alpha}{\beta}P\right)z(t)\leq 1.$$

Theorem is proven. ■

15.7 Matrix Inequality Simplification

Introduce the following orthogonal matrix

$$G_B \in \mathbb{R}^{n\times n},\ G_B G_B^{\intercal} = I_{n\times n},\ G_B B = \begin{pmatrix} 0_{(n-m)\times m} \\ I_{m\times m} \end{pmatrix} \tag{15.32}$$

The corresponding matrix G_B can be easily found in MATLAB using the function *null*. For given matrix M the function null(M) returns the matrix that columns are orthonormal basis of the null space of the matrix M, namely,

$$G_B = \begin{pmatrix} B^{\perp} \\ B' \end{pmatrix},$$

where

$$B^{\perp} = [\text{null}(B^{\intercal})]^{\intercal},\ B' = \left[\text{null}(B^{\perp})\right]^{\intercal}.$$

Since the descriptive multipliers Π_a and Π_b, may be any $(2n\times 2n)$ - matrices, select them as

$$\Pi_a = \Pi_b = P\tilde{G}_B, \tag{15.33}$$

where

$$\tilde{G}_B := \begin{bmatrix} G_B & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

satisfying

$$\begin{gathered}\tilde{G}_B\tilde{G}_B^{\intercal} = I_{2n\times 2n},\ \tilde{G}_B B_0 = \begin{bmatrix} G_B & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} \end{bmatrix}\begin{bmatrix} B & 0_{n\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix} \\ = \begin{bmatrix} \begin{bmatrix} 0_{(n-m)\times m} \\ I_{m\times m} \end{bmatrix} & 0_{n\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix} = \begin{bmatrix} 0_{(n-m)\times(n+m)} \\ I_{(n+m)\times(n+m)} \end{bmatrix}.\end{gathered}$$

Then for

$$P = \begin{bmatrix} P_1 & 0_{(n-m)\times(n+m)} \\ 0_{(n+m)\times(n-m)} & P_2 \end{bmatrix},$$

$$P_1 \in \mathbb{R}^{(n-m)\times(n-m)},\ P_2 \in \mathbb{R}^{(n+m)\times(n+m)}$$

it follows

$$P\tilde{G}_B B_0 \Theta C_0 = \begin{bmatrix} 0_{(n-m)\times(n+m)} \\ YC_0 \end{bmatrix},\ YC_0 \in \mathbb{R}^{(n+m)\times(n+m)}$$

$$P\tilde{G}_B B_0 \Theta E_0 = \begin{bmatrix} 0_{(n-m)\times(n+m)} \\ YE_0 \end{bmatrix},\ YE_0 \in \mathbb{R}^{(n+m)\times(n+m)}$$

where

$$\boxed{Y = P_2\Theta,} \tag{15.34}$$

and defining

$$\boxed{X = P = \begin{bmatrix} X_1 & 0_{(n-m)\times(n+m)} \\ 0_{(n+m)\times(n-m)} & X_2 \end{bmatrix},\ X_1 = P_1,\ X_2 = P_2,} \tag{15.35}$$

as the result, for the transforming matrix

$$T = \operatorname{diag}\left(\tilde{G}_B,\ I_{2n\times 2n},\ I_{2n\times 2n},\ I_{(n+2l)\times(n+2l)}\right)$$

we get

$$\boxed{\begin{array}{c} W_{\tilde{G}_B}(X, Y \mid \alpha) := TWT^{\intercal} = \\ \\ \begin{bmatrix} W_{\tilde{G}_B}^{(1,1)} & W_{\tilde{G}_B}^{(1,2)} & 0 & W_{\tilde{G}_B}^{(1,4)} \\ \left(W_{\tilde{G}_B}^{(1,2)}\right)^{\intercal} & W_{\tilde{G}_B}^{(2,2)} & 0 & W_{\tilde{G}_B}^{(2,4)} \\ 0 & 0 & W_{\tilde{G}_B}^{(3,3)} & 0 \\ \left(W_{\tilde{G}_B}^{(1,4)}\right)^{\intercal} & \left(W_{\tilde{G}_B}^{(2,4)}\right)^{\intercal} & 0 & W_{\tilde{G}_B}^{(4,4)} \end{bmatrix} < 0 \end{array},} \tag{15.36}$$

where

$$W_{\tilde{G}_B}^{(1,1)} = \alpha X + M^{\intercal} Q_x M + X\tilde{G}_B A_0 + A_0^{\intercal}\left(\tilde{G}_B\right)^{\intercal} X + \begin{bmatrix} 0_{(n-m)\times(n+m)} \\ YC_0 \end{bmatrix} + \begin{bmatrix} 0_{(n-m)\times(n+m)} \\ YC_0 \end{bmatrix}^{\intercal},$$

$$W_{\tilde{G}_B}^{(1,2)} = X\left(I_{2n\times 2n} + \tilde{G}_B A_0\right) + \begin{bmatrix} 0_{(n-m)\times(n+m)} \\ YC_0 \end{bmatrix},$$

$$W_{\tilde{G}_B}^{(1,4)} = W_{\tilde{G}_B}^{(2,4)} = X\tilde{G}_B D_0 + \begin{bmatrix} 0_{(n-m)\times(n+m)} \\ YE_0 \end{bmatrix},\ W_{\tilde{G}_B}^{(4,4)} = -Q,$$

$$W_{\tilde{G}_B}^{(2,2)} = h^2 R - X\tilde{G}_B - \left(\tilde{G}_B\right)^{\intercal} X,\ W_{\tilde{G}_B}^{(3,3)} = -he^{-\alpha h} R + 4M^{\intercal} C^{\intercal} Q_y CM.$$

Notice that (15.36) is a linear form with respect to the matrix variables X and Y .

15.8 Optimal Feedback Parameters

If an attractive ellipsoid $\mathcal{E}(0, P_{attr})$ with

$$P_{attr} = \frac{\beta}{\alpha} P$$

has the smallest size (we mean the trace of the associated ellipsoid matrix P_{attr}^{-1}), then it seems to be natural to call the corresponding parameters Θ^* of the designed robust feedback *optimal*. Introducing the upper matrix estimate

$$H > X^{-1},$$

that can be equivalently represented (by the Schur inequality) as

$$\begin{bmatrix} H & I_{2n\times 2n} \\ {}_{2n\times 2n} & X \end{bmatrix} > 0, \tag{15.37}$$

we may define the optimal parameters Θ^* of the dynamic controller as the solution to the following matrix optimization problem

$$\operatorname{tr}\left(\frac{\alpha}{\beta} X^{-1}\right) \leq \operatorname{tr}\left(\frac{\alpha}{\beta} H\right) \to \min_{X>0,\ H>0,\ Y,\ \alpha>0} \tag{15.38}$$

subject the matrix constraints (15.36) and (15.37)

$$W_{\tilde{G}_B}(X, Y \mid \alpha) < 0, \quad \begin{bmatrix} H & I_{2n\times 2n} \\ I_{2n\times 2n} & X \end{bmatrix} > 0$$

such that if X^* and Y^* are the solutions of the problem (15.38), then by (15.34) and (15.35)

$$\boxed{\Theta^* = (X_2^*)^{-1} Y^*.} \tag{15.39}$$

15.8.1 Example

Example 15.1 *Consider the dynamic plant governed by*

$$\left.\begin{aligned} \dot{x}_1(t) &= 0.5\sqrt{|x_1(t)|}\operatorname{sign}(x_1(t)) + u + \xi_1(t), \\ \dot{x}_2(t) &= \sin(x_1(t)) + u + \xi_2(t), \\ y(t) &= x_1(t) + 0.15\sin(5t), \ t \geq 0 \\ x_1(0) &= -1.5, \ x_2(0) = -2 \end{aligned}\right\} \tag{15.40}$$

where $\xi_1(t)$, $\xi_2(t)$ are independent "white" noise (generated by the Simulink tool-box) with the amplitude $M = 0.2$, the output is quantized with interval $c = 0.05$, the maximal sample-time is $h = 0.2$. Here in the format (15.1) - (15.2)

$$B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We also selected

$$A = \begin{bmatrix} -10 & -2 \\ -5 & -2 \end{bmatrix}, \; \delta = 1, \; Q = I_{2\times 2}, \; \beta = 9 + \delta + 2c = 10.01,$$

and the matrix R in (15.18) as

$$R = \begin{bmatrix} 25 & 21 & 0 & 0 \\ 21 & 21.512 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5.2808 \end{bmatrix}$$

As the result, we obtained

$$\alpha^* = 0.1562$$

and the solution of the optimization problem (15.38) is

$$X^* = \begin{bmatrix} 1.1147 & 0 & 0 & 0 \\ 0 & 1.1871 & 0 & 0 \\ 0 & 0 & 1.1419 & 0 \\ 0 & 0 & 0 & 1.2971 \end{bmatrix}, \; Y^* = \begin{bmatrix} -0.9016 & 0 & -1 \\ 0.5 & -1 & 0 \\ 0.5 & 0 & -1 \end{bmatrix},$$

$$(X_2^*)^{-1} = \begin{bmatrix} 1.1871 & 0 & 0 \\ 0 & 1.1419 & 0 \\ 0 & 0 & 1.2971 \end{bmatrix}, \; H^* = \begin{bmatrix} 1.5637 & 0 & 0 & 0 \\ 0 & 1.1594 & 0 & 0 \\ 0 & 0 & 1.6786 & 0 \\ 0 & 0 & 0 & 1.1778 \end{bmatrix}.$$

$$\Theta^* = (X_2^*)^{-1} Y^* = \begin{bmatrix} D_r^* & C_r^* \\ B_r^* & A_r^* \end{bmatrix} = \begin{bmatrix} -0.7587 & 0 & -0.8419 \\ 0.4389 & -0.8752 & 0 \\ 0.3861 & 0 & -0.7733 \end{bmatrix}.$$

$$D_r^* = -0.7587, \; C_r^* = \begin{bmatrix} 0, & -0.8419 \end{bmatrix}, \; B_r^* = \begin{bmatrix} 0.4389 \\ 0.3861 \end{bmatrix}, \; A_r^* = \begin{bmatrix} -0.8752 & 0 \\ 0 & -0.7733 \end{bmatrix}.$$

The figurs below illustrate the process (see Figues 15.3 - 15.5) controlled under obtained values of the feedback.

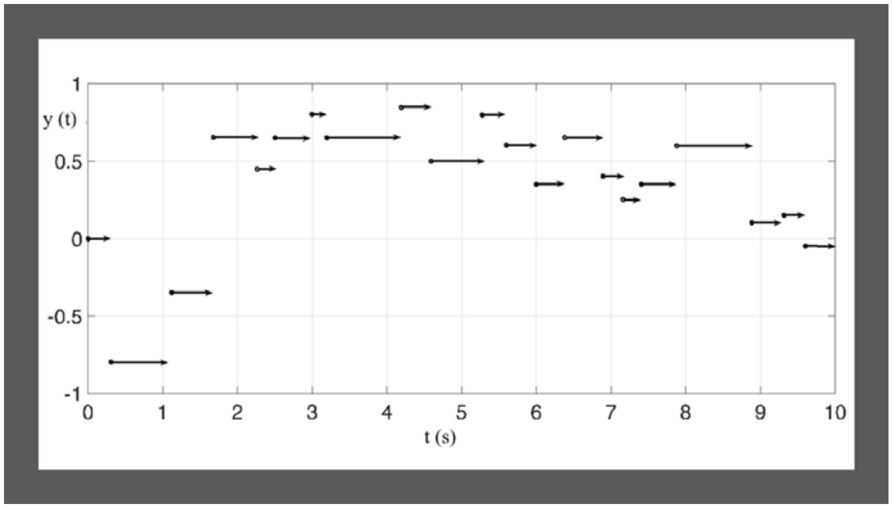

Figure 15.2: Measurable output $y(t)$.

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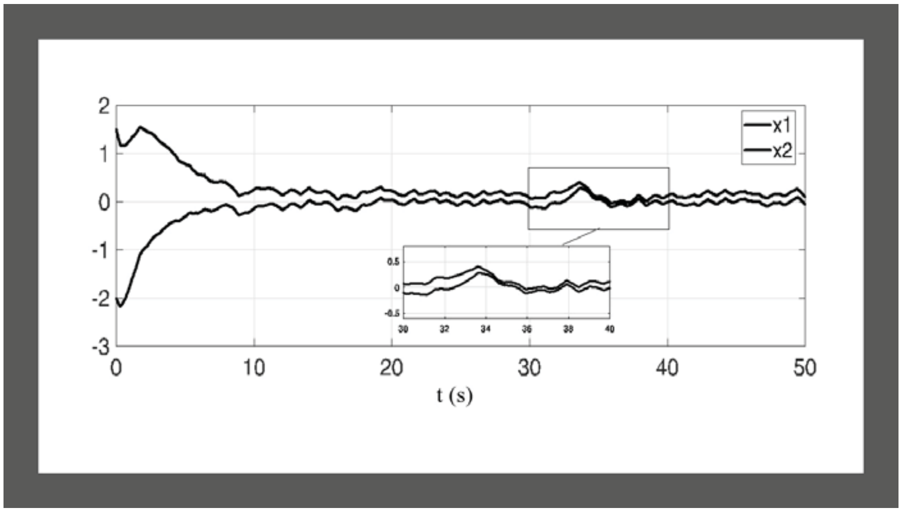

Figure 15.3: Controlled trajectories $x_1(t)$, $x_2(t)$.

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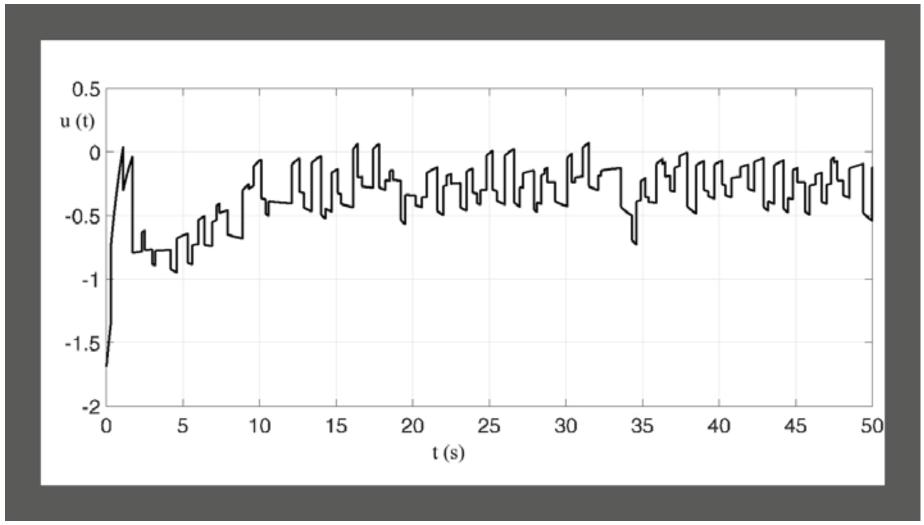

Figure 15.4: Robust control action $u(t)$.

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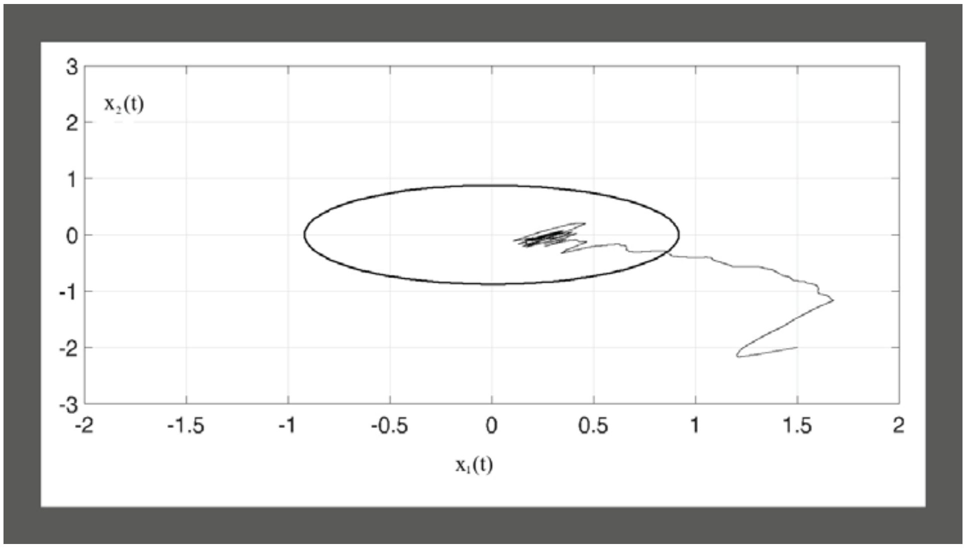

Figure 15.5: Convergence of trajectories to the attractive ellipsoid.

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